DISCOVERY

Integral points on the homogeneous cone z² = 5x² + 11y²

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ABSTRACT

The homogeneous cone represented by the ternary quadratic Diophantine equation $z^2 = 5x^2 + 11y^2$ is analyzed for its patterns of non zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Homogeneous cone, Ternary quadratic, Integral solutions.

1. INTRODUCTION

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety. For an extensive review of various problems, one may refer (Dickson et al, 1952, Mordell et al, 1969, Gopalan et al, 2000a & 2000b). This communication concerns with yet another interesting Ternary Quadratic Equation $z^2 = 5x^2 + 11y^2$ representing a homogeneous cone for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented (Dickson et al, 1952, Mordell et al, 1969, Gopalan et al, 2000c).

2. NOTATION

tm,n= polygonal number of rank r with sides m.

3. METHOD OF ANALYSIS

The homogeneous cone represented by the ternary quadratic equation is

$$z^2 = 5x^2 + 11y^2 \qquad (1)$$

It is observed that (1) is satisfied by infinitely many non-zero distinct integral solutions. For the sake of clear understanding, we present below different patterns of solutions to (1).

3.1. Pattern 1
Introducing the linear transformations
$$x = X + 11T \; ; \; y = X - 5T \; ; \; z = 4W \qquad (2)$$

in (1), it is written as
$$W^2 = X^2 + 55T^2 \qquad (3)$$

which is satisfied by

$$T = 2rs \; ; X = 55r^2 - s^2; W = 55r^2 + s^2 \tag{4}$$

From (4) and (2), the non-zero integral solutions of (1) are given by

$$x = 55r^2 - s^2 + 22rs \; ; y = 55r^2 - s^2 - 10rs \; ; \; z = 220r^2 + 4s^2$$

A few interesting properties satisfied by the above solutions are given below:

(i) $x(r,1) - 110t_{3,r} \equiv -1 (mod\ 33)$

(ii) $x(r,3) - 108t_{3,r} - t_{4,r+6} = -45$

(iii) $y(r,4) - 10t_{13,r} - t_{4,r+3} - 2t_{3,r} = -25$

(iv) $x(r,1) - 33t_{4,r} - 44t_{3,r} = -1$

(v)) $x(r,2) - 90t_{3,r} - 5t_{6,r} \equiv 0 (mod\ 4)$

(vi) $y(r,4) - 10t_{13,r}$ is a perfect square for the values of r given by (a) $r = 5n^2 - 6n + 5$

(b) $20n^2 - 28n + 13$ (c) $25n^2 - 23n + 14$.

3.2. Pattern 2
Instead of (2) we may consider the linear transformations

$$x = X - 11T \; ; y = X + 5T \; ; z = 4W \tag{5}$$

Substituting (4) in (5), the integral solutions to (1) are obtained as

$$x = 55r^2 - s^2 - 22rs \; ; y = 55r^2 - s^2 + 10rs \; ; \; z = 220r^2 + 4s^2$$

Some of the interesting properties of the above equations are

(i) $y(r,1) - 110t_{3,r} \equiv -1 (mod\ 45)$

(ii) $x(r,1) - 110t_{3,r} \equiv -1 (mod\ 77)$

(iii) $x(r,2) - 11t_{12,r} = -4$

Remark
It is worth mentioning here that (3) may also be solved by applying the method of factorization. After performing some algebra, we obtain the following triples of integers satisfying (1):

$(38,22,112)\,,(16,32,112)\,,(14,-2,32)\,,(-8,8,32)\,,(2a^2 + 24a - 16, 2a^2 - 8a - 32, 8a^2 + 8a + 112)\,,(2a^2 - 20a - 38, 2a^2 +$

$12a - 22, 8a^2 + 8a + 112)\,,(22a^2 - 8, 22a^2 + 32a + 8, 88a^2 + 88a + 32)\,,(22a^2 + 44a + 14, 22a^2 + 12a - 2, 88a^2 + 88a +$

$32)\,,(10a^2 + 32a + 8, 10a^2 - 8, 40a^2 + 40a + 32)\,,(10a^2 - 12a - 14, 10a^2 + 20a + 2, 40a^2 + 40a + 32)$.

3.3. Pattern 3
Rewrite (3) as

$$X^2 + 55T^2 = W^2 = W^2 * 1 \tag{6}$$

Assume $W = a^2 + 55b^2 \tag{7}$

Write 1 as $1 = \frac{(3+i\sqrt{55})(3-i\sqrt{55})}{64}$ (8)

Using (7) and (8) in (6) and applying the method of factorization, define

$$X + i\sqrt{55}T = (a + i\sqrt{55}b)^2 \frac{(3 + i\sqrt{55})}{8}$$

Equating the real and imaginary parts, we get

$$\left. \begin{array}{l} X = \frac{1}{8}(3a^2 - 165b^2 - 110ab) \\[6pt] T = \frac{1}{8}(a^2 - 55b^2 + 6ab) \end{array} \right\} \qquad (9)$$

Using (7) and (9) in (2), the integral solutions to (1) are obtained as

$$x = -a^2 + 55b^2 - 22ab \; ; y = a^2 - 55b^2 - 10ab \; ; z = 4a^2 + 220b^2$$

A few properties of the above solutions are

(i) $x(R(R + 1), 1) + y(R(R + 1), 1) = -64t_{3,R}$

(ii) $z(R(R + 1), 1) - y(R(R + 1), 1) - 2t_{5,R(R+1)} - 11t_{4,R(R+1)+5} + 22t_{3,R(R+1)} = -176t_{3,R}$

(iii) $y(R(R + 1), 2) + x(R(R + 1), 1) + 84t_{3,R} = -165$

(iv) $x(R(R + 1), 2) + y(R(R + 1), 2) = -128t_{3,R}$

(v) $x(r, 1) + 4t_{3,r} - t_{4,r-10} = -45$

(vi) $z(r, 1) - 220t_{4,r} = 4$

(vii) $y(r, 1) - 24t_{3,r} + 11t_{4,r+1} = -44$

Also, using (7) and (9) in the alternate transformations (5), the corresponding solutions to (1) are given by

$$x = \frac{1}{4}(7a^2 - 385b^2 - 22ab)$$

$$y = \frac{1}{4}(-a^2 + 55b^2 - 70ab)$$

$$z = 4a^2 + 220b^2$$

As we are interested in the integer solutions, note that x and y are integers for suitable choices of a and b. We present below the choices of a and b along with the corresponding solutions to (1)

Choice 1: Let $a = 2A$, $b = 2B$

The corresponding solutions to (1) are

$$x = 7A^2 - 385B^2 - 22AB$$

$$y = -A^2 + 55B^2 - 70AB$$

$$z = 16A^2 + 880B^2$$

Choice 2: Let $a = (2k - 1)b$

The corresponding solutions to (1) are given by

$$x = 7b^2k^2 - 18b^2k - 89b^2$$

$$y = -b^2k^2 - 34b^2k + 31b^2$$

$$z = 16b^2k^2 - 16b^2k + 224b^2$$

Choice 3: Let $b = (2k - 1)a$

The corresponding integral solutions to (1) are obtained as

$$x = -385a^2k^2 + 374a^2k - 89a^2$$

$$y = 55a^2k^2 - 90a^2k + 31a^2$$

$$z = 880a^2k^2 - 880a^2k + 224a^2$$

3.4. Pattern 4

Rewrite (3) as

$$W^2 - 55T^2 = X^2 = X^2 * 1 \tag{10}$$

Assume $X = a^2 - 55b^2$ $\tag{11}$

Write 1 as $1 = \dfrac{(8+\sqrt{55})(8-\sqrt{55})}{9}$ $\tag{12}$

Using (11) and (12) in (10) and applying the method of factorization, define

$$W + \sqrt{55}T = (a + \sqrt{55}b)^2 \frac{(8+\sqrt{55})}{3}$$

Equating the rational and irrational parts, we obtain

$$\left.
\begin{aligned}
W &= \tfrac{1}{3}(8a^2 + 110ab + 440b^2) \\
T &= \tfrac{1}{3}(a^2 + 16ab + 55b^2)
\end{aligned}
\right\} \tag{13}$$

Using (11) and (13) in (2), the solutions to (1) are obtained as

$$\left.
\begin{aligned}
x &= \tfrac{1}{3}(14a^2 + 176ab + 440b^2) \\
y &= \tfrac{1}{3}(-2a^2 - 80ab - 440b^2) \\
z &= \tfrac{4}{3}(8a^2 + 110ab + 440b^2)
\end{aligned}
\right\} \tag{14}$$

It can be noted that the above solution (14) is not an integer solution. In order to obtain integral solutions, we employ some

substitutions which are illustrated below:

$$\text{(i) Let } a = 3b, b = 3a$$

The corresponding solutions to (1) are obtained as

$$x = 42b^2 + 528ab + 1320a^2$$

$$y = -6b^2 - 240ab - 1320a^2$$

$$z = 96b^2 + 1320ab - 5280a^2$$

(ii) Let $a = (3k + 1)b$

The corresponding solutions to (1) are

$$x = 42b^2k^2 + 204b^2k + 210b^2$$

$$y = -6b^2k^2 - 84b^2k - 174b^2$$

$$z = 96b^2k^2 + 504b^2k + 744b^2$$

Also, using (11) and (13) in (5), we obtain the solutions to (1) as

$$\left. \begin{aligned} x &= \tfrac{1}{3}(-8a^2 - 176ab - 770b^2) \\ y &= \tfrac{1}{3}(8a^2 + 80ab + 110b^2) \end{aligned} \right\} \tag{15}$$

$$z = \frac{4}{3}(8a^2 + 110ab + 440b^2)$$

The above solution (15) is not an integral solution. To obtain the integral solution, we impose some substitutions which are

exhibited below:

(i) Let $a = 3b$, $b = 3a$

The corresponding solution to (1) is given by

$$x = -24b^2 - 528ab - 2310a^2$$

$$y = 24b^2 + 240ab + 330a^2$$

$$z = 96b^2 + 1320ab + 5280a^2$$

(ii) Let $a = (3k + 1)b$

The corresponding integral solutions to (1) are

$$x = -24b^2k^2 - 192b^2k - 318b^2$$

$$y = 24b^2k^2 + 96b^2k + 66b^2$$

$$z = 96b^2k^2 + 504b^2k + 744b^2$$

(iii) Let $b = (6k + 1)a$

The corresponding integral solutions to (1) are

$$x = -9240a^2k^2 - 3432a^2k - 318a^2$$

$$y = 1320a^2k^2 + 600a^2k + 66a^2$$

$$z = 63360a^2k^2 + 7920a^2k + 744a^2$$

4. REMARKABLE OBSERVATIONS

If the non-zero integer triple (x_0, y_0, z_0) is the initial solution of (1), then each of the following triples of integers also satisfy (1).

4.1. Triple 1: (x_0, y_n, z_n)

where $\alpha = 10 + 3\sqrt{11}$, $\beta = 10 - 3\sqrt{11}$

$$y_n = \frac{1}{2\sqrt{11}}[\sqrt{11}(\alpha^n + \beta^n)y_0 + (\alpha^n - \beta^n)z_0]$$

$$z_n = \frac{1}{2\sqrt{11}}[11(\alpha^n - \beta^n)y_0 + \sqrt{11}(\alpha^n + \beta^n)z_0]$$

4.2 Triple 2: (x_n, y_0, z_n)

where $\alpha = 9 + 4\sqrt{5}$, $\beta = 9 - 4\sqrt{5}$

$$x_n = \frac{1}{2\sqrt{5}}[\sqrt{5}(\alpha^n + \beta^n)x_0 + (\alpha^n - \beta^n)z_0]$$

$$z_n = \frac{1}{2\sqrt{5}}[5(\alpha^n - \beta^n)x_0 + \sqrt{5}(\alpha^n + \beta^n)z_0]$$

4.3 Triple 3: (x_n, y_n, z_n)

where $\alpha = 8$, $\beta = -8$

$$x_n = \frac{1}{16}[(5\alpha^n + 11\beta^n)x_0 + 11(\alpha^n - \beta^n)y_0]$$

$$y_n = \frac{1}{16}[5(\alpha^n - \beta^n)x_0 + (11\alpha^n + 5\beta^n)y_0]$$

$$z_n = 8^n z_0$$

5. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES AND NOTES

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